Lecture 10: Dynamic Factor Models II

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Outline

- In the previous lecture we developed methodology for dynamic factor models
- In this set of slides we show how factor models can be used for structural analyses
- In some sense we simply combine factor models with SVAR models

Recall SVAR

Summary of reduced and structural forms

Reduced form

$$\mathbf{y}_t = \mathbf{A}_1\mathbf{y}_{t-1} + \ldots + \mathbf{A}_p\mathbf{y}_{t-p} + \boldsymbol{u}_t \qquad \boldsymbol{u}_t \sim WN(0, \boldsymbol{\Sigma}_u)$$

$$\mathbf{y}_t = \sum_{i=0}^{\infty} \boldsymbol{\Phi}_i \boldsymbol{u}_{t-i}$$

Structural form

$$\mathbf{B}_0\mathbf{y}_t = \mathbf{B}_1\mathbf{y}_{t-1} + \ldots + \mathbf{B}_p\mathbf{y}_{t-p} + \boldsymbol{w}_t \qquad \boldsymbol{w}_t \sim WN(0, \mathbf{I}_K)$$

$$\mathbf{y}_t = \sum_{i=0}^{\infty} \boldsymbol{\Theta}_i \boldsymbol{w}_{t-i}$$

Relationship

$$\boldsymbol{u}_t = \mathbf{B}_0^{-1}\boldsymbol{w}_t \qquad \mathbf{A}_i = \mathbf{B}_0^{-1}\mathbf{B}_i \qquad \boldsymbol{\Theta}_i = \boldsymbol{\Phi}_i\mathbf{B}_0^{-1}$$

Short run restrictions

Restrict elements of $\mathbf{B}_0^{-1}$ such that you can solve

$$\mathbf{\Sigma}_u = \mathbf{B}_0^{-1}\mathbf{B}_0^{-1\prime}$$

Implementation

- Estimate the reduced form coefficients $\hat{\mathbf{A}}(L), \hat{\mathbf{\Sigma}}_u$
- Solve $\hat{\mathbf{\Sigma}}_u = \hat{\mathbf{B}}_0^{-1}\hat{\mathbf{B}}_0^{-1\prime}$
- Construct the impulse responses using $\hat{\mathbf{\Theta}}_i = \hat{\mathbf{\Phi}}_i\hat{\mathbf{B}}_0^{-1}$

Identification using external instruments

- Assume you have a scalar instrument z_t and that the structural shock of interest is the first shock $w_{1,t}$

Key Assumptions

1. $\mathbb{E}(z_t w_{1,t}) = c \neq 0$ (Relevant)
2. $\mathbb{E}(z_t w_{j,t}) = 0$ for all $j \neq 1$ and t (Exogeneity)

Further, because the scales of $\boldsymbol{u}_t = \mathbf{B}_0^{-1}\boldsymbol{w}_t$ is not uniquely pinned down we restrict $b_0^{11} = 1$

Identifying the impulse response

Let $\mathbf{B}_{0,1}^{-1}$ denote the first column of $\mathbf{B}_0^{-1}$

$$\mathbf{\Theta}_{i,1} = \mathbf{\Phi}_i \mathbf{B}_{0,1}^{-1},$$

To get this column note that

$$\mathbf{\Gamma} = \mathbb{E}(z_t \boldsymbol{u}_t) = \mathbb{E}(z_t \mathbf{B}_0^{-1} \boldsymbol{w}_t) = c\mathbf{B}_{0,1}^{-1}$$

which identifies the column up to scale and using $b_0^{11} = 1$ we obtain

$$\mathbf{\Theta}_{i,1} = \mathbf{\Phi}_i \mathbf{\Gamma} / \mathbf{\Gamma}_{11}$$

Estimation

Replace population quantities by sample estimates

$$\hat{\boldsymbol{\Theta}}_{i,1} = \hat{\boldsymbol{\Phi}}_i \hat{\boldsymbol{\Gamma}} / \hat{\boldsymbol{\Gamma}}_1$$

where $\hat{\boldsymbol{\Gamma}} = \frac{1}{T}\sum_{t=1}^{T} z_t \hat{\mathbf{u}}_t$, where $\hat{\mathbf{u}}_t$ is the reduced form residual.

- The strength of the instrument can be assessed using the F-statistic for the first stage regression of z_t on $\hat{\mathbf{u}}_t$
- Weak inference robust inference is discussed in Olea, Stock & Watson (2018)

Motivation

Motivation

Why combine factor models with SVAR models?

- To capture omitted variables in small scale SVARs
- To capture measurement error in small scale SVARs

Invertibility in VAR models

For the SVAR model the structural MA representation is given by

$$\mathbf{Y}_t = \sum_{i=0}^{\infty} \mathbf{\Theta}_i \boldsymbol{w}_{t-i}$$

SVAR models typically assume that $\boldsymbol{u}_t = \mathbf{B}_0^{-1} \boldsymbol{w}_t$ such that

$$\boldsymbol{w}_t = \mathbf{B}_0 \mathbf{A}(L) \mathbf{Y}_t$$

- If this is true we say that the SVAR model is invertible or fundamental
- In words: the structural shocks can be recovered from the observable $\mathbf{Y}_t$

Invertibility in VAR models

$$\boldsymbol{w}_t = \mathbf{B}_0 \mathbf{A}(L) \mathbf{Y}_t$$

Breaks down if we have omitted variables

- For example, suppose that there are four shocks of interest (monetary policy, productivity, demand, oil supply) but only three variables (interest rates, GDP, the oil price) in the VAR.
- It is impossible to reconstruct the four shocks from current and lagged values of the three observed time series, so the structural moving average process is not invertible.
- Estimates from a SVAR constructed from the VAR innovations will therefore suffer from a form of omitted variable bias.

Invertibility in VAR models

$$\boldsymbol{w}_t = \mathbf{B}_0\mathbf{A}(L)\mathbf{Y}_t$$

Breaks down if we have measurement error

- Some elements of $\mathbf{Y}_t$ may be measured with error, which effectively adds more shocks (the measurement error) to the model.
- Again, this makes it impossible to reconstruct the structural shocks from current and lagged values of $\mathbf{Y}_t$.
- This source of non-invertibility can be thought of as errors-in-variables bias.

Invertibility in VAR models

- Omitted variables and measurement error can be captured by augmenting the SVAR model with the dynamic factor model
- Two different (albeit closely related) approaches are
  - Structual Dynamic Factor Models (SDFM)
  - or Factor Augmented Vector Autoregressive Models (FAVAR)

Structural Dynamic Factor Models

Structural Dynamic Factor Model

$$\mathbf{Y}_t = \boldsymbol{\Lambda} \boldsymbol{f}_t + \epsilon_t$$

$$\mathbf{A}(L)\boldsymbol{f}_t = \boldsymbol{u}_t \qquad \boldsymbol{u}_t = \mathbf{B}_0^{-1}\boldsymbol{w}_t$$

where

- The reduced form shocks $\boldsymbol{u}_t$ are mapped to the structural shocks $\boldsymbol{w}_t$
- The rest of the model is identical to the dynamic factor model of the previous lecture
- It is possible to have less structural shocks then reduced form shocks, we omit this extension, see Stock & Watson (2016) for details

Structural Dynamic Factor Model

The moving average representation for f_t can be written as

$$\boldsymbol{f}_t = \sum_{i=0}^{\infty} \boldsymbol{\Theta}_i \boldsymbol{w}_{t-i} \qquad \boldsymbol{\Theta}_i = \boldsymbol{\Phi}_i \mathbf{B}_0^{-1}$$

which we can plug into the observation equation to get

$$\mathbf{Y}_t = \boldsymbol{\Lambda} \sum_{i=0}^{\infty} \boldsymbol{\Theta}_i \boldsymbol{w}_{t-i} + \boldsymbol{\epsilon}_t$$

Now the impulse response of a one-unit change in $\boldsymbol{w}_t$ is given by

$$\frac{\partial \mathbf{Y}_{t+h}}{\partial \boldsymbol{w}_t'} = \boldsymbol{\Lambda}\boldsymbol{\Theta}_i$$

Structural Dynamic Factor Model

$$\mathbf{Y}_t = \boldsymbol{\Lambda} \boldsymbol{f}_t + \epsilon_t$$

$$\mathbf{A}(L)\boldsymbol{f}_t = \boldsymbol{u}_t \qquad \boldsymbol{u}_t = \mathbf{B}_0^{-1}\boldsymbol{w}_t$$

The SDFM requires two sets of restrictions

- Statistic identification restrictions that identify the common factor structure
- Structural identification restrictions that identify the structural shocks

Importantly, interpretation is requires close alignment of the statistical and structural choices

Structural Dynamic Factor Model

It is important to use named factors

$$\boldsymbol{\Lambda} = \begin{bmatrix} \mathbf{I}_r \\ \boldsymbol{\Lambda}_2 \end{bmatrix}$$

This directly maps the common factors to the first r observables and thus facilitates interpretation

- You might worry then this excludes PCA (as it imposes a different identification restriction) but recall that we can always rotate the factors, such that

$$\hat{\boldsymbol{\Lambda}} = \begin{bmatrix} \mathbf{I}_r \\ \hat{\boldsymbol{\Lambda}}_2^{PCA} \left(\hat{\boldsymbol{\Lambda}}_1^{PCA}\right)^{-1} \end{bmatrix} \qquad \hat{\mathbf{F}} = \hat{\boldsymbol{\Lambda}}_1^{PCA} \hat{\mathbf{F}}^{PCA}$$

Structural Dynamic Factor Model

- This approach is easily extended to handle multiple observable for the same factor
- Example, suppose that we have 4 oil prices $y_{1,t}^{PPI}$, $y_{2,t}^{Brent}$, $y_{3,t}^{WTI}$ and $y_{4,t}^{RAC}$ we may consider

$$\begin{bmatrix} y_{1,t}^{PPI} \\ y_{2,t}^{Brent} \\ y_{3,t}^{WTI} \\ y_{4,t}^{RAC} \\ \mathbf{Y}_{5:N,t} \end{bmatrix} = \begin{bmatrix} 1 & 0 \dots & 0 \\ 1 & 0 \dots & 0 \\ 1 & 0 \dots & 0 \\ 1 & 0 \dots & 0 \\ & \boldsymbol{\Lambda}_{5:N} & \end{bmatrix} \begin{bmatrix} f_{1,t}^{oil} \\ \boldsymbol{f}_{2:r,t} \end{bmatrix} + \boldsymbol{\epsilon}_t$$

Structural Dynamic Factor Model

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Structural Dynamic Factor Model

- Given that the factors have been named the identification methods for the structural shocks from Lecture 6 carry over
- Error bands can be computed using bootstrap methods for the entire SDFM
- A parametric bootstrap example is given in Stock & Watson (2016) on page 476

FAVAR models

FAVAR models

- A slightly different approach for combining factor model and SVAR models is developed by Bernanke, Boivin & Eliasz (2005)
- They develop the Factor Augmented VAR (FAVAR)
- FAVARs model some of the factors as observed variables while the remaining factors are unobserved

FAVAR models

For simplicity, we consider the FAVAR model with one observed factor $\tilde{f}_t$

$$\begin{bmatrix} x_t \\ \mathbf{Y}_t \end{bmatrix} = \begin{bmatrix} 1 \; 0_{1\times r} \\ \boldsymbol{\Lambda} \end{bmatrix} \begin{bmatrix} \tilde{f}_t \\ \boldsymbol{f}_t \end{bmatrix} + \begin{bmatrix} 0 \\ \epsilon_t \end{bmatrix}$$

$$\boldsymbol{f}_t^+ = \begin{bmatrix} \tilde{f}_t \\ \boldsymbol{f}_t \end{bmatrix} \qquad \mathbf{A}(L)\boldsymbol{f}_t^+ = \boldsymbol{u}_t \qquad \boldsymbol{u}_t = \mathbf{B}_0^{-1}\boldsymbol{w}_t$$

- In total the model has $r+1$ factors
- The first factor is observed as the variable x_t

FAVAR models

We can rewrite the FAVAR model by substituting $\tilde{f}_t = x_t$ to obtain

$$\mathbf{Y}_t = \mathbf{\Lambda} \begin{bmatrix} x_t \\ \boldsymbol{f}_t \end{bmatrix} + \epsilon_t$$

$$\boldsymbol{f}_t^+ = \begin{bmatrix} x_t \\ \boldsymbol{f}_t \end{bmatrix} \qquad \mathbf{A}(L)\boldsymbol{f}_t^+ = \boldsymbol{u}_t \qquad \boldsymbol{u}_t = \mathbf{B}_0^{-1}\boldsymbol{w}_t$$

- This rewrite shows that the FAVAR identification problem is the same as the SVAR identification problem

FAVAR models

To illustrate lets consider the original Bernanke et al application

- The goal is to identify the effects of monetary policy shocks using short run restrictions
- This original FAVAR application achieves two goals
  - First, by including a large number of variables, it addresses the omitted variable problem of low-dimensional VARs and in particular aims to resolve the so-called 'price puzzle' of monetary VARs (see Ramey, 2016)
  - Second, the joint modeling of these many variables permits estimating internally consistent structural IRFs for an arbitrarily large list of variables of interest

FAVAR models

Let $\mathbf{Y}_t^s$ and $\mathbf{Y}_t^f$ denote slow and fast moving variables, we have

$$\begin{bmatrix} \mathbf{Y}_t^s \\ \mathbf{Y}_t^f \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Lambda}_{ss} & \mathbf{0} & \mathbf{0} \\ \boldsymbol{\Lambda}_{fs} & \boldsymbol{\Lambda}_{fr} & \boldsymbol{\Lambda}_{ff} \end{bmatrix} \begin{bmatrix} \boldsymbol{f}_t^s \\ r_t \\ \boldsymbol{f}_t^f \end{bmatrix} + \boldsymbol{\epsilon}_t$$

$$\boldsymbol{f}_t^+ = \begin{bmatrix} \boldsymbol{f}_t^s \\ r_t \\ \boldsymbol{f}_t^f \end{bmatrix} \qquad \mathbf{A}(L)\boldsymbol{f}_t^+ = \boldsymbol{u}_t$$

$$\begin{bmatrix} \boldsymbol{u}_t^s \\ u_t^r \\ \boldsymbol{u}_t^f \end{bmatrix} = \begin{bmatrix} b_0^{11} & 0 & 0 \\ b_0^{21} & b_0^{22} & 0 \\ b_0^{31} & b_0^{32} & b_0^{33} \end{bmatrix} \begin{bmatrix} \boldsymbol{w}_t^s \\ w_t^r \\ \boldsymbol{w}_t^f \end{bmatrix}$$

Impulse responses: Bernanke et al

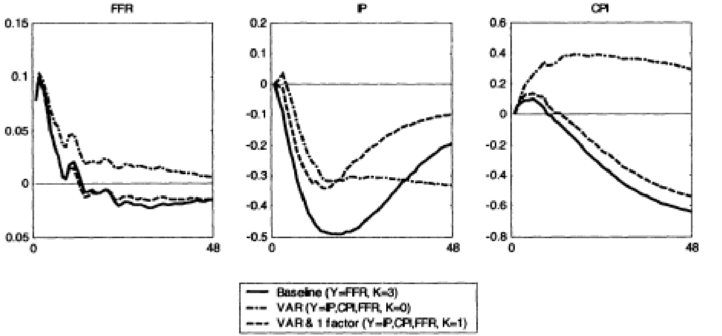

FIGURE I
Estimated Impulse Responses to an Identified Policy Shock for Alternative FAVAR Specifications, Based on the Two-Step Principal Component's Approach

References & Material

Reference & Material

- References:
  J. Stock & M. Watson (2016) Dynamic Factor Models, Factor-Augmented Vector Autoregressions, and Structural Vector Autoregressions in Macroeconomics, in Chapter 8 of Handbook of Macroeconomics, Volume 2A