Lecture 8: State Space Models: part II

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Motivation

- In the previous lecture we illustrated the class of state space models via the local level model
- In this set of slides we show that the general class of state space models includes many popular models encountered in time series analysis
- Further, we discuss multiple empirical examples from macroeconomics where state space models are useful

A brief recap

Local level model

Recal

$$\begin{aligned} y_t &= \alpha_t + \epsilon_t & \epsilon_t &\sim N(0, \sigma_\epsilon^2) \\ \alpha_{t+1} &= \alpha_t + \eta_t & \eta_t &\sim N(0, \sigma_\eta^2) \end{aligned}$$

- y_t only observable
- Goal is to learn about the latent state α_t

Kalman filter

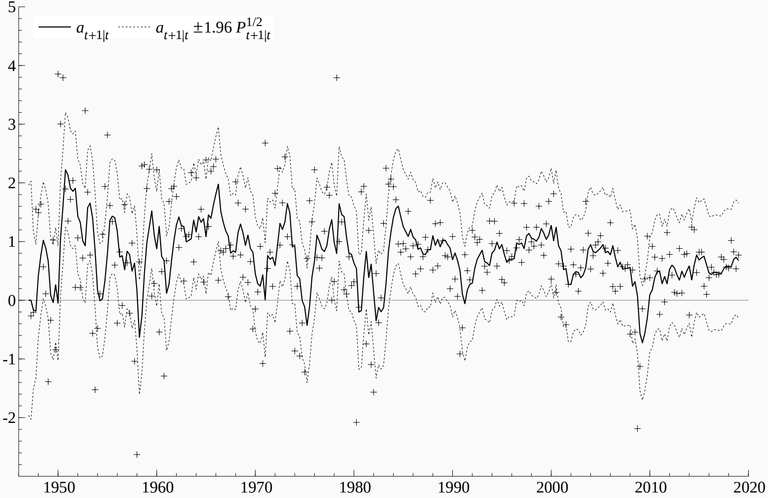
Theorem

The predictive and filtered estimates are computed by the Kalman filter for the local level model as follows:

$$
\begin{aligned}
v_t &= y_t - a_{t|t-1} & F_t &= P_{t|t-1} + \sigma_\epsilon^2 \\
a_{t|t} &= a_{t|t-1} + K_t v_t & P_{t|t} &= P_{t|t-1}(1 - K_t) \\
a_{t+1|t} &= a_{t|t} & P_{t+1|t} &= P_{t|t} + \sigma_\eta^2
\end{aligned}
$$

for $t = 1, \ldots, T$ *and* $K_t = P_{t|t-1}/F_t$.

Filtered US GDP



Backward smoothing recursions

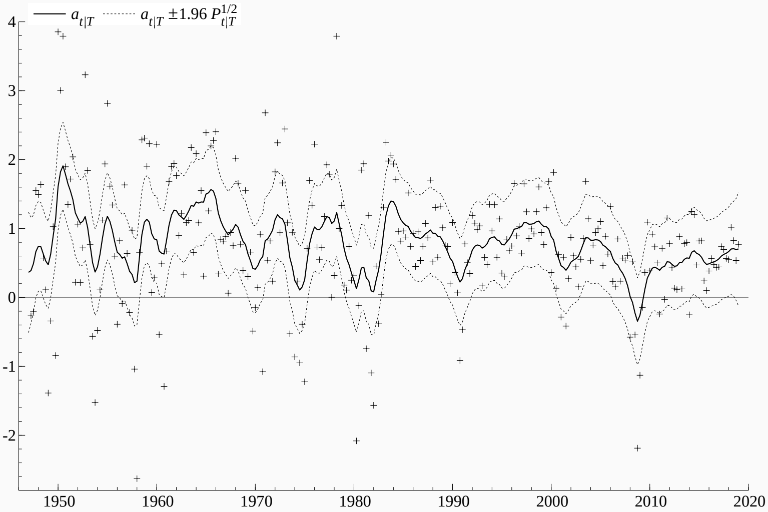
Theorem

The smoothed estimates are computed by the backward smoothing recursions for the local level model as follows:

$$
\begin{aligned}
a_{t|T} &= a_{t|t-1} + P_{t|t-1} r_{t-1} & \quad r_{t-1} &= F_t^{-1} v_t + L_t r_t \\
P_{t|T} &= P_{t|t-1} - P_{t|t-1}^2 N_{t-1} & \quad N_{t-1} &= F_t^{-1} + L_t^2 N_t
\end{aligned}
$$

for $t = T, \ldots, 1$ *and* $L_t = 1 - K_t$ *and* $K_t = P_{t|t-1} / F_t$.

Smoothed US GDP



Further

- We could estimate deterministic parameters by maximum likelihood
- Missing values and forecasts were handled using the Kalman filter by setting the Kalman gain equal to zero
- Initialization was done either diffuse or unconditional
- Errors can and should be checked

Class of state space models

Linear Gaussian state space model

The linear Gaussian state space model that we consider is given by

$$\begin{aligned} \boldsymbol{Y}_t &= \boldsymbol{Z}_t \boldsymbol{\alpha}_t + \boldsymbol{\epsilon}_t & \boldsymbol{\epsilon}_t &\sim N(\mathbf{0}, \boldsymbol{H}_t) \\ \boldsymbol{\alpha}_{t+1} &= \boldsymbol{A}_t \boldsymbol{\alpha}_t + \boldsymbol{R}_t \boldsymbol{\eta}_t & \boldsymbol{\eta}_t &\sim N(\mathbf{0}, \boldsymbol{Q}_t) \end{aligned}$$

where $\boldsymbol{Y}_t$ is the $m \times 1$ vector of observations and $\boldsymbol{\alpha}_t$ is the $l \times 1$ state vector.

- The system matrices $\boldsymbol{Z}_t$, $\boldsymbol{H}_t$, $\boldsymbol{A}_t$, $\boldsymbol{R}_t$ and $\boldsymbol{Q}_t$ may depend on unknown parameters.
- We first discuss each equation separately

Observation equation

$$\boldsymbol{Y}_t = \boldsymbol{Z}_t \boldsymbol{\alpha}_t + \boldsymbol{\epsilon}_t \qquad \epsilon_t \sim N(\mathbf{0}, \boldsymbol{H}_t)$$

- $\boldsymbol{Z}_t$ is the $m \times l$ matrix that links the state equations to the observations.
- The measurement error ϵ_t is assumed serially uncorrelated and normally distributed.
- Intuitively: the observations are a deterministic combination of l unobserved stochastic processes which are contaminated with measurement error

State equation

$$\boldsymbol{\alpha}_{t+1} = \boldsymbol{A}_t \boldsymbol{\alpha}_t + \boldsymbol{R}_t \boldsymbol{\eta}_t \qquad \boldsymbol{\eta}_t \sim N(\mathbf{0}, \boldsymbol{Q}_t)$$

- The state equation follows a vector autoregressive model of order 1
- $\boldsymbol{A}_t$ is the $l \times l$ transition matrix that links the future state to the current state
- The $l \times r$ matrix $\boldsymbol{R}_t$ links the $r \times 1$ state disturbances $\boldsymbol{\eta}_t$ to the state equation.
- The state disturbances are serially uncorrelated and independent from the observation disturbances.

Linear Gaussian state space model

The linear Gaussian state space model with dimensions in brackets

$$\begin{aligned}
\boldsymbol{Y}_t^{(m\times 1)} &= \boldsymbol{Z}_t^{(m\times l)}\boldsymbol{\alpha}_t^{(l\times 1)} + \boldsymbol{\epsilon}_t^{(m\times 1)} & \boldsymbol{\epsilon}_t \sim N(\boldsymbol{0}^{(m\times 1)}, \boldsymbol{H}_t^{(m\times m)}) \\
\boldsymbol{\alpha}_{t+1}^{(l\times 1)} &= \boldsymbol{A}_t^{(l\times l)}\boldsymbol{\alpha}_t^{(l\times 1)} + \boldsymbol{R}_t^{(l\times r)}\boldsymbol{\eta}_t^{(r\times 1)} & \boldsymbol{\eta}_t \sim N(\boldsymbol{0}^{(r\times 1)}, \boldsymbol{Q}_t^{(r\times r)})
\end{aligned}$$

- The component $\boldsymbol{Z}_t\boldsymbol{\alpha}_t$ is referred to as the signal and $\boldsymbol{\epsilon}_t$ as the noise component
- The flexibility in the dimensions allows us to write many models in this form

Main results

Main results

In the next slides we state the main algorithms for handling the general linear Gaussian state space model.

- Kalman filter
- Smoothing recursions
- Log likelihood

Derivations can be found in Durbin & Koopman (2012)

Kalman filter

Theorem

Kalman filter for the Linear Gaussian State Space Model

$$\begin{aligned}
\boldsymbol{v}_t &= \boldsymbol{Y}_t - \boldsymbol{Z}_t \boldsymbol{a}_{t|t-1} \\
\boldsymbol{F}_t &= \boldsymbol{Z}_t \boldsymbol{P}_{t|t-1} \boldsymbol{Z}_t' + \boldsymbol{H}_t \\
\boldsymbol{a}_{t|t} &= \boldsymbol{a}_{t|t-1} + \boldsymbol{P}_{t|t-1} \boldsymbol{Z}_t' \boldsymbol{F}_t^{-1} \boldsymbol{v}_t \\
\boldsymbol{P}_{t|t} &= \boldsymbol{P}_{t|t-1} - \boldsymbol{P}_{t|t-1} \boldsymbol{Z}_t' \boldsymbol{F}_t^{-1} \boldsymbol{Z}_t \boldsymbol{P}_{t|t-1} \\
\boldsymbol{a}_{t+1|t} &= \boldsymbol{A}_t \boldsymbol{a}_{t|t-1} + \boldsymbol{K}_t \boldsymbol{v}_t \\
\boldsymbol{P}_{t+1|t} &= \boldsymbol{A}_t \boldsymbol{P}_{t|t-1} (\boldsymbol{A}_t - \boldsymbol{K}_t \boldsymbol{Z}_t)' + \boldsymbol{R}_t \boldsymbol{Q}_t \boldsymbol{R}_t'
\end{aligned}$$

for $t = 1, \ldots, T$, *where* $\boldsymbol{K}_t = \boldsymbol{A}_t \boldsymbol{P}_{t|t-1} \boldsymbol{Z}_t' \boldsymbol{F}_t^{-1}$.

Smoothing recursions

Theorem

State smoothing for the Linear Gaussian State Space Model

$$\begin{aligned}
\hat{\boldsymbol{\alpha}}_t &= \boldsymbol{a}_{t|t-1} + \boldsymbol{P}_{t|t-1}\boldsymbol{r}_{t-1} \\
\boldsymbol{V}_t &= \boldsymbol{P}_{t|t-1} - \boldsymbol{P}_{t|t-1}\boldsymbol{N}_{t-1}\boldsymbol{P}_{t|t-1} \\
\boldsymbol{r}_{t-1} &= \boldsymbol{Z}_t'\boldsymbol{F}_t^{-1}\boldsymbol{v}_t + \boldsymbol{L}_t'\boldsymbol{r}_t \\
\boldsymbol{N}_{t-1} &= \boldsymbol{Z}_t'\boldsymbol{F}_t^{-1}\boldsymbol{Z}_t + \boldsymbol{L}_t'\boldsymbol{N}_t\boldsymbol{L}_t
\end{aligned}$$

for $t = T, T-1, \ldots, 1$, *where* $\boldsymbol{L}_t = \boldsymbol{A}_t - \boldsymbol{K}_t\boldsymbol{Z}_t$.

Maximum likelihood estimation

- The parameters of the Linear Gaussian state space model, summarized in ψ, can be estimated by maximum likelihood

The log likelihood is given by

$$\ell(\boldsymbol{Y}_{1:T};\psi) = \text{constant} - \frac{1}{2}\sum_{t=1}^{T}\left(\log|\boldsymbol{F}_t| + \boldsymbol{v}_t'\boldsymbol{F}_t^{-1}\boldsymbol{v}_t\right)$$

and the maximum likelihood estimates are given by

$$\hat{\psi} = \arg\max_{\psi}\ell(\boldsymbol{Y}_{1:T};\psi)$$

Examples state space representation

To illustrate the flexibility of state space models we consider

- Unobserved components models
- Multivariate unobserved components models
- (V)ARMA(p,q) models
- Time-varying parameter regression model
- Dynamic factor models (next lectures)

Unobserved components models

Unobserved components models

The class of unobserved components models is given by

$$y_t = \mu_t + \gamma_t + c_t + \epsilon_t$$

where

- y_t scalar observation
- μ_t trend component
- γ_t seasonal component
- c_t cycle component
- ϵ_t idiosyncratic component

The idea is to decompose the variable Y_t into trend, seasonal, cycle and idiosyncratic components. You can view this class of models as an alternative to the ARMA model class where the variables explained in terms of past observations and shocks.

Local Level Model

An important member of this class is the local level model

$$\begin{aligned} y_t &= \mu_t + \epsilon_t & \epsilon_t &\sim N(0, \sigma_\epsilon^2) \\ \mu_{t+1} &= \mu_t + \eta_t & \eta_t &\sim N(0, \sigma_\eta^2) \end{aligned}$$

where the observations are modeled by a random walk trend and a noise component. The state space representation is obtained by setting $\alpha_t = \mu_t$ and the system matrices are taken as

$$\begin{aligned} \boldsymbol{Z}_t &= 1 & \boldsymbol{H}_t &= \sigma_\epsilon^2 \\ \boldsymbol{A}_t &= 1 & \boldsymbol{R}_t &= 1 & \boldsymbol{Q}_t &= \sigma_\eta^2 \end{aligned}$$

Local Linear Trend Model

We can include a slope in the trend component to obtain the local linear trend model

$$
\begin{aligned}
y_t &= \mu_t + \epsilon_t & \epsilon_t &\sim N(0, \sigma^2_\epsilon) \\
\mu_{t+1} &= \mu_t + \beta_t + \eta_t & \eta_t &\sim N(0, \sigma^2_\eta) \\
\beta_{t+1} &= \beta_t + \xi_t & \xi_t &\sim N(0, \sigma^2_\xi)
\end{aligned}
$$

The state space representation is obtained by setting $\boldsymbol{\alpha}_t = (\mu_t, \beta_t)'$ and the system matrices are taken as

$$
\begin{aligned}
\boldsymbol{Z}_t &= [1, 0] & \boldsymbol{H}_t &= \sigma^2_\epsilon \\
\boldsymbol{A}_t &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} & \boldsymbol{R}_t &= \boldsymbol{I}_2 & \boldsymbol{Q}_t &= \begin{bmatrix} \sigma^2_\eta & 0 \\ 0 & \sigma^2_\xi \end{bmatrix}
\end{aligned}
$$

Local Linear Trend Model + seasonal component

Consider the following seasonal component for s seasonal periods

$$\gamma_{t+1} = -\sum_{j=1}^{s-1} \gamma_{t+1-j} + \omega_t \qquad \omega_t \sim N(0, \sigma_\omega^2)$$

which allows for a seasonal pattern that may vary across time due to shocks ω_t. Note that if $\omega_t = 0$ we simply have seasonal dummies. Adding this to the local linear trend model gives

$$\begin{array}{rcl} y_t &=& \mu_t + \gamma_t + \epsilon_t \\ \mu_{t+1} &=& \mu_t + \beta_t + \eta_t \\ \beta_{t+1} &=& \beta_t + \xi_t \\ \gamma_{t+1} &=& -\sum_{j=1}^{s-1} \gamma_{t+1-j} + \omega_t \end{array} \qquad \begin{array}{rcl} \epsilon_t &\sim& N(0, \sigma_\epsilon^2) \\ \eta_t &\sim& N(0, \sigma_\eta^2) \\ \xi_t &\sim& N(0, \sigma_\xi^2) \\ \omega_t &\sim& N(0, \sigma_\omega^2) \end{array}$$

Local Linear Trend Model + seasonal component

The state space representation is obtained by setting $\boldsymbol{\alpha}_t = (\mu_t, \beta_t, \gamma_t, \gamma_{t-1}, \ldots, \gamma_{t-s+2})'$ and the system matrices are taken as

$$\boldsymbol{Z}_t = [1, 0, 1, 0 \ldots, 0] \qquad \boldsymbol{H}_t = \sigma^2_\epsilon$$

$$\boldsymbol{A}_t = \begin{bmatrix} 1 & 1 & 0 & \ldots & \ldots & \ldots & 0 \\ 0 & 1 & -1 & -1 & \ldots & -1 & -1 \\ 0 & 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & 1 & \ldots & 0 & 0 \\ & & & & \ddots & & \\ 0 & 0 & 0 & 0 & \ldots & 1 & 0 \end{bmatrix} \qquad \boldsymbol{R}_t = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix}$$

$$\boldsymbol{Q}_t = \begin{bmatrix} \sigma^2_\eta & 0 & 0 \\ 0 & \sigma^2_\xi & 0 \\ 0 & 0 & \sigma^2_\omega \end{bmatrix}$$

Unobserved components and US GDP growth

- Next, we use the unobserved components approach to decompose US GDP
- In the previous lecture we used a local level model to illustrate the approach
- We now extend our efforts to include trend, cycle and idiosyncratic components

Decomposing US GDP

Consider

$$y_t = \mu_t + c_t + \epsilon_t$$

where c_t is the stochastic cycle component that is modeled as

$$\begin{bmatrix} c_t \\ c_t^* \end{bmatrix} = \rho \begin{bmatrix} \cos\omega_c & \sin\omega_c \\ -\sin\omega_c & \cos\omega_c \end{bmatrix} \begin{bmatrix} c_{t-1} \\ c_{t-1}^* \end{bmatrix} + \begin{bmatrix} u_t \\ u_t^* \end{bmatrix}$$

with

- ω_c the frequency (treated as unknown parameter)
- u_t, u_t^* are independent normals with variance σ_u^2

A word on stochastic cycles

$$\begin{bmatrix} c_t \\ c_t^* \end{bmatrix} = \rho \begin{bmatrix} \cos\omega_c & \sin\omega_c \\ -\sin\omega_c & \cos\omega_c \end{bmatrix} \begin{bmatrix} c_{t-1} \\ c_{t-1}^* \end{bmatrix} + \begin{bmatrix} u_t \\ u_t^* \end{bmatrix}$$

Let's write out the equation

$$c_t = \rho\cos(\omega_c)c_{t-1} + \rho\sin(\omega_c)c_{t-1}^* + u_t$$

which is a weighted average of a sine and cosine.

- Weights are stochastic and time-varying
- Only one frequency matters whose importance may fluctuate over time

Trend cycle model in state space form

We can write the model in the general state space form

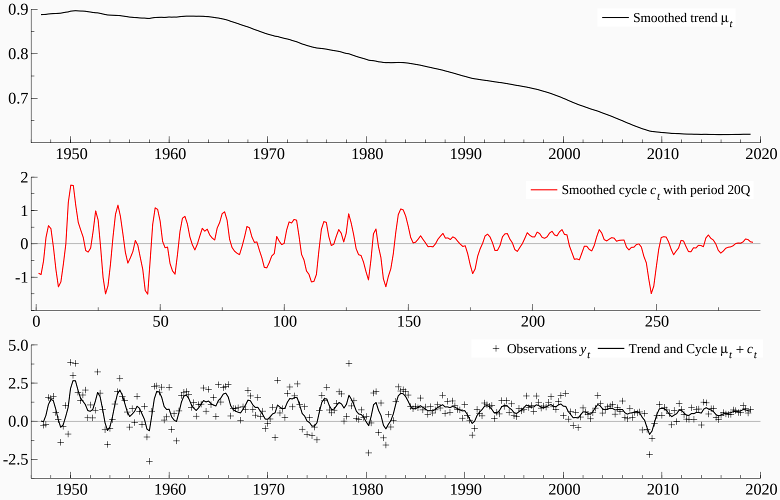
$$\boldsymbol{Z}_t = [1, 1, 0] \qquad \boldsymbol{H}_t = \sigma_\epsilon^2$$

$$\boldsymbol{A}_t = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\omega_c) & \sin(\omega_c) \\ 0 & -\sin(\omega_c) & \cos(\omega_c) \end{bmatrix}$$

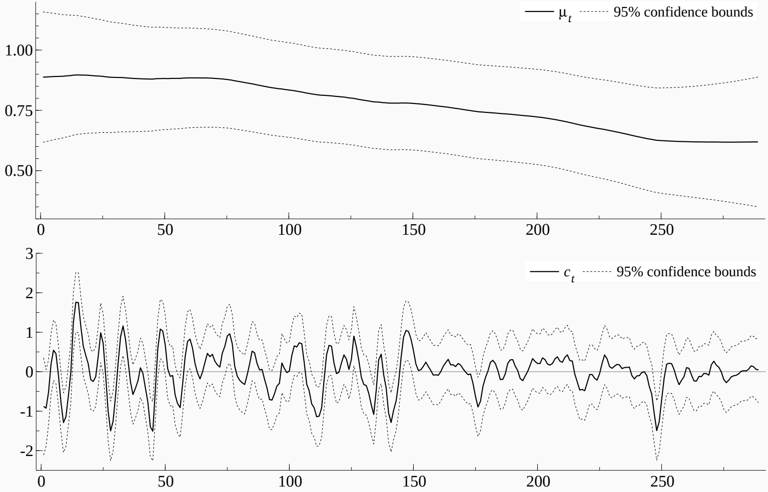
and

$$\boldsymbol{R}_t = \boldsymbol{I}_3 \qquad \boldsymbol{Q}_t = \text{diag}(\sigma_\mu^2, \sigma_u^2, \sigma_u^2)$$

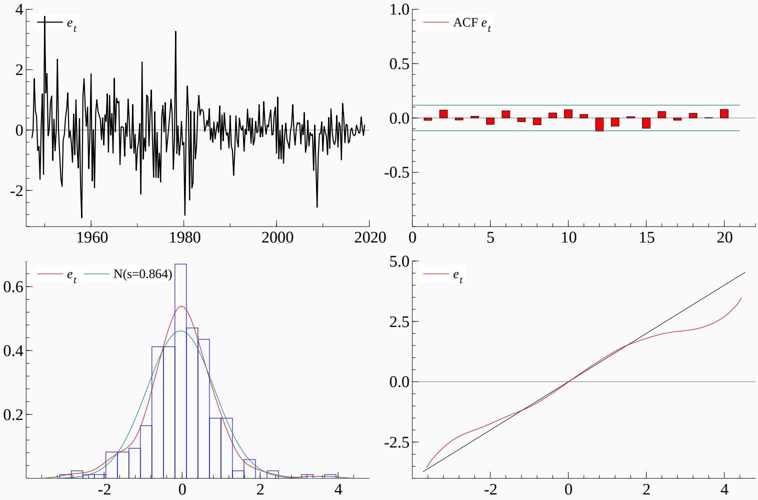
Trend cycle model for US GDP



Trend cycle model for US GDP



Trend cycle model for US GDP



Multivariate unobserved components models

Multivariate unobserved components models

Unobserved components models can be naturally extended to multivariate settings

$$\boldsymbol{Y}_t = \boldsymbol{Z}\boldsymbol{\mu}_t + \boldsymbol{\epsilon}_t$$

where

- $\boldsymbol{Y}_t$ $m \times 1$ observations
- $\boldsymbol{\mu}_t$ $l \times 1$ trend component
- $\boldsymbol{Z}$ $m \times l$ transition matrix
- $\boldsymbol{\epsilon}_t$ $m \times 1$ idiosyncratic component

Other components can be added in the same way as before. We illustrate the multivariate version by estimating the US and World business cycle.

Business conditions measurement

- Using a small scale dynamic factor model Arouba, Diebold & Scotti propose to measure business conditions
- They use a 1 state model which they estimate using maximum likelihood methods
- They incorporate several time series that are measured at different intervals

Mixed frequency data

1. initial jobless claims (weekly)
2. payroll employment (monthly)
3. industrial production (monthly)
4. real personal income (monthly)
5. real manufacturing and trade sales (monthly)
6. real GDP (quarterly)

These six series are used to construct the current index that is available from the Philadelphia FED.

Model specifics

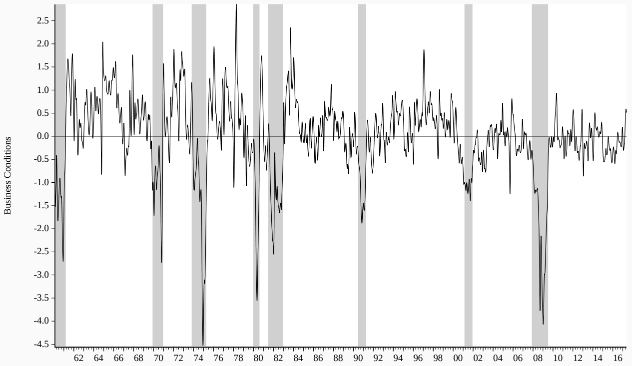
$$\begin{aligned} \boldsymbol{Y}_t &= \boldsymbol{Z}\mu_t + X_t\beta + \epsilon_t & \epsilon_t &\sim NID(0, \boldsymbol{H}) \\ \mu_t &= a_1\mu_{t-1} + \ldots + a_p\mu_{t-p} + \eta_t & \eta_t &\sim NID(0, \sigma_\eta) \end{aligned}$$

where

- μ_t is scalar trend that measures business conditions
- X_t includes a variety of time dummies and trends

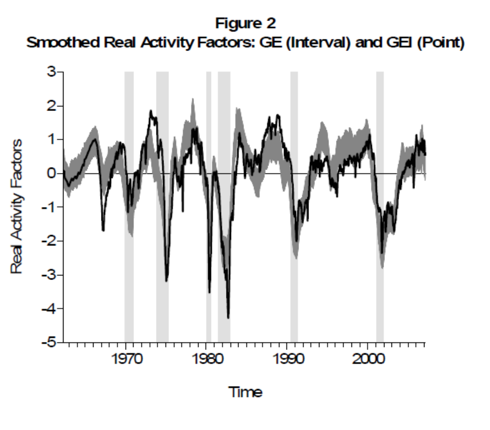
Extracted common trend

Aruoba-Diebold-Scotti Business Conditions Index (3/1/1960 - 05/13/2017)

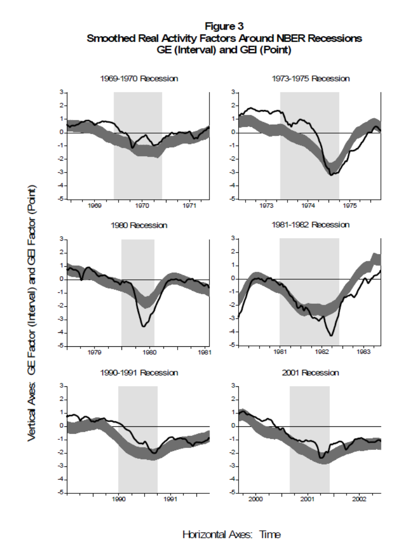


Note: We construct the ADS Index using the latest data available as of May 18, 2017. This includes (1) initial jobless claims through the week ending May 13, 2017, (2) payroll employment through April 2017, (3) industrial production through April 2017, (4) real personal income through March 2017, (5) real manufacturing and trade sales through February 2017, and (6) real GDP through the first quarter of 2017. Gray shading indicates NBER-designated recessions. The limits used on the y axis reflect the minimum and maximum values of the index over its entire history.

Benefit from high frequency data



Benefit from high frequency data around recession



Measuring world business cycle

- Since the mid-1980s many countries have experienced similar fluctuations in output and other macro-economic aggregates.
- This has raised the question whether the business cycles of different countries are converging.
- A number of papers have empirically studied business cycles across countries; see Kose, Otrok & Whiteman (2003).

Data

- $\boldsymbol{Y}_t$ includes the economic growth rates for $N = 101$ countries listed in the Penn World Tables (PTW) between 1950 and 2011
- The resulting panel is highly unbalanced. For example, in the initial year 1950 there are 49 missing observations whereas in 2011 there are observations recorded for all 101 countries.
- The different countries can be split into 7 regions and our goal is to separate global growth from regional growth rates

Model specification

$$\begin{aligned}\boldsymbol{Y}_t &= \boldsymbol{Z}\boldsymbol{\mu}_t + \boldsymbol{\epsilon}_t & \boldsymbol{\epsilon}_t &\sim NID(0, \boldsymbol{H}) \\ \boldsymbol{\mu}_{t+1} &= \boldsymbol{A}\boldsymbol{\mu}_t + \boldsymbol{\eta}_t & \boldsymbol{\eta}_t &\sim NID(0, \boldsymbol{Q})\end{aligned}$$

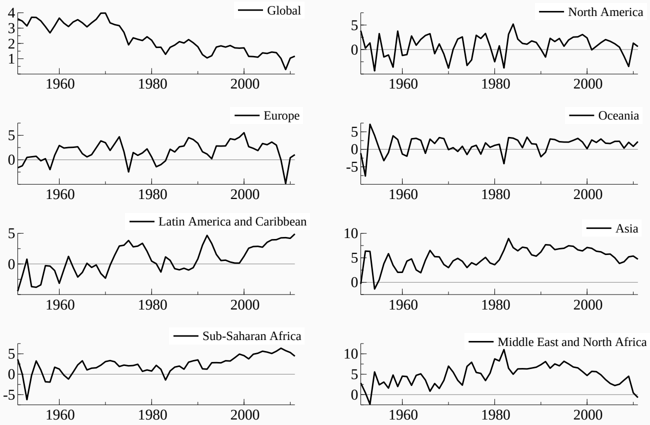
where

- $\boldsymbol{\mu}_t$ includes 8 stationary trend components
- To separate world and regional conditions we allocate the trends as follows. Suppose that country i is in region 3. The mapping of country i becomes

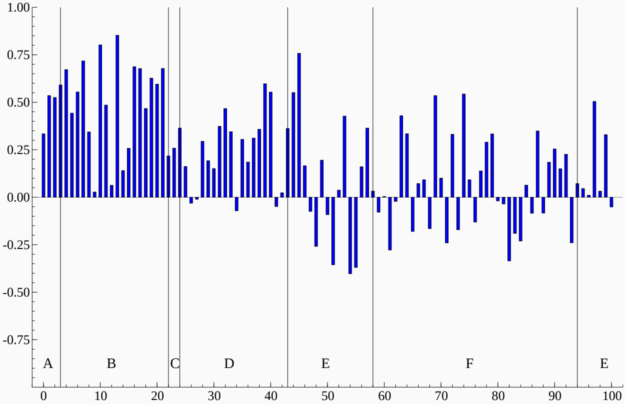
$$\boldsymbol{z}_i = (\lambda_{i,world}, 0, 0, \lambda_{i,region3}, 0, 0, 0, 0)$$

- Hence all countries load on the world trend but each country only load on 1 regional trend
- The model parameters are estimated by maximum likelihood

Trends

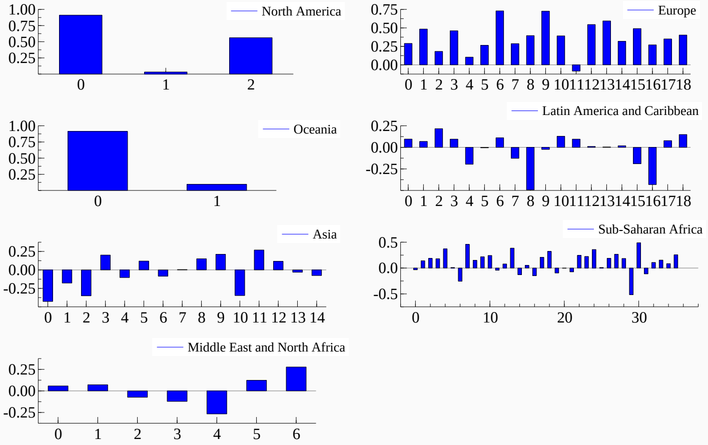

Global
North America
Europe
Oceania
Latin America and Caribbean
Asia
Sub-Saharan Africa
Middle East and North Africa
1960
1980
2000

Correlations with the world[1]



[1] A = North America, B = Europe, C = Oceania, D = Latin America and the Caribbean, E = Asia, F = Sub-Saharan Africa and G = Middle East and North Africa.

Per country correlation with regions



(V)ARMA models

ARMA(p,q) models: ARMA(1,0)

All ARMA(p,q) models can be also written in state space form. We show a few examples and then give the general form. The ARMA(1,0) model is given by

$$Y_{t+1} = \phi_1 Y_t + \zeta_t \qquad \zeta_t \sim N(0, \sigma_\zeta^2)$$

which we can write in state space form by taking $\alpha_t = Y_t$ and taking the system matrices

$$\begin{array}{lllllllll} \boldsymbol{Z}_t & = & 1 & \boldsymbol{H}_t & = & 0 & & & \\ \boldsymbol{A}_t & = & \phi_1 & \boldsymbol{R}_t & = & 1 & \boldsymbol{Q}_t & = & \sigma_\zeta^2 \end{array}$$

where we notice that the dynamics of the ARMA model are modeled in the state equation.

ARMA(p,q) models: ARMA(2,1)

The ARMA(2,1) model is given by

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \zeta_t + \theta_1 \zeta_{t-1} \qquad \zeta_t \sim N(0, \sigma^2_\zeta)$$

which we can write in state space form by taking $\boldsymbol{\alpha}_t = (Y_t, \phi_2 Y_{t-1} + \theta_1 \zeta_t)'$ and taking the system matrices

$$\begin{aligned} \boldsymbol{Z}_t &= [1, 0] & \boldsymbol{H}_t &= 0 & & \\ \boldsymbol{A}_t &= \begin{bmatrix} \phi_1 & 1 \\ \phi_2 & 0 \end{bmatrix} & \boldsymbol{R}_t &= \begin{bmatrix} 1 \\ \theta_1 \end{bmatrix} & \boldsymbol{Q}_t &= \sigma^2_\zeta \end{aligned}$$

Check for yourself that the state equation is correct for this choice of state and system matrices

ARMA(p,q) models: ARMA(p,q)

The ARMA(p,q) model can be written as

$$Y_t = \sum_{j=1}^{k} \phi_j Y_{t-j} + \zeta_t + \sum_{j=1}^{k-1} \theta_j \zeta_{t-j} \qquad \zeta_t \sim N(0, \sigma_\zeta^2)$$

where

- $k = \max(p, q+1)$
- Some of the coefficients ϕ_j and θ_j may be zero depending on the orders p and q
- We introduce this general notation to obtain a simple state space notation

ARMA(p,q) models: ARMA(p,q)

The state space form is obtained by taking the state equation as

$$\boldsymbol{\alpha}_t = \begin{pmatrix} Y_t \\ \phi_2 Y_{t-1} + \cdots + \phi_k Y_{t-k+1} + \theta_1 \zeta_t + \cdots + \theta_{k-1} \zeta_{t-k+2} \\ \phi_3 Y_{t-1} + \cdots + \phi_k Y_{t-k+2} + \theta_2 \zeta_t + \cdots + \theta_{k-1} \zeta_{t-k+3} \\ \vdots \\ \phi_k Y_{t-1} + \theta_{k-1} \zeta_t \end{pmatrix}$$

where the second to last terms are merely included for correct updating.

ARMA(p,q) models: ARMA(p,q)

The system matrices are given by

$$
\begin{aligned}
\boldsymbol{Z}_t &= [1, 0, \ldots, 0] & \boldsymbol{H}_t &= 0 \\
\boldsymbol{A}_t &= \begin{bmatrix} \phi_1 & 1 & & 0 \\ \vdots & & \ddots & \\ \phi_{k-1} & 0 & & 1 \\ \phi_k & 0 & \ldots & 0 \end{bmatrix} & \boldsymbol{R}_t &= \begin{bmatrix} 1 \\ \theta_1 \\ \vdots \\ \theta_{k-1} \end{bmatrix} \\
\boldsymbol{Q}_t &= \sigma_\zeta^2
\end{aligned}
$$

Check again that this formulation is correct by writing out the state equation.

VARMA(p,q) models

In the same way we can consider VARMA(p,q) models. In general notation we can write these as

$$\boldsymbol{Y}_{t+1} = \sum_{j=1}^{k} \boldsymbol{\Phi}_j \boldsymbol{Y}_{t-j} + \boldsymbol{\zeta}_t + \sum_{j=1}^{k-1} \boldsymbol{\Theta}_j \boldsymbol{\zeta}_{t-j} \qquad \boldsymbol{\zeta}_t \sim N(\mathbf{0}, \boldsymbol{\Sigma}_\zeta)$$

where

- $\boldsymbol{Y}_{t+1}$ is the $m \times 1$ vector of time series
- $\boldsymbol{\Phi}_j$ and $\boldsymbol{\Theta}_j$ are the $m \times m$ coefficient matrices of which some may be zero
- The state space form is the same as for the ARMA(p,q) model with the coefficients ϕ_j and θ_j replaced appropriately by the matrices $\boldsymbol{\Phi}_j$ and $\boldsymbol{\Theta}_j$

Time-varying parameter regression

TVP regression models

Consider the regression model with time varying parameters for Y_t with k regressors $\boldsymbol{X}_t$

$$
\begin{aligned}
Y_t &= \boldsymbol{X}_t' \boldsymbol{\beta}_t + \epsilon_t & \epsilon_t &\sim N(0, \sigma_\epsilon^2) \\
\boldsymbol{\beta}_{t+1} &= \boldsymbol{\beta}_t + \boldsymbol{\eta}_t & \boldsymbol{\eta}_t &\sim N(0, \boldsymbol{\Sigma}_\eta)
\end{aligned}
$$

This model can be written in state space form by taking $\boldsymbol{\alpha}_t = \boldsymbol{\beta}_t$ and taking the system matrices

$$
\begin{aligned}
\boldsymbol{Z}_t &= \boldsymbol{X}_t & \boldsymbol{H}_t &= \sigma_\epsilon^2 & & \\
\boldsymbol{A}_t &= \boldsymbol{I}_k & \boldsymbol{R}_t &= \boldsymbol{I}_k & \boldsymbol{Q}_t &= \boldsymbol{\Sigma}_\eta
\end{aligned}
$$

This is a useful model if you want to study changes in the parameter estimates.

Slope of Phillips curve

- We illustrate the time-varying parameter regression model by studying whether the slope of the Phillips curve has declined over time
- See Ball & Mazumder (2012) and Blanchard (2016) for some evidence of this

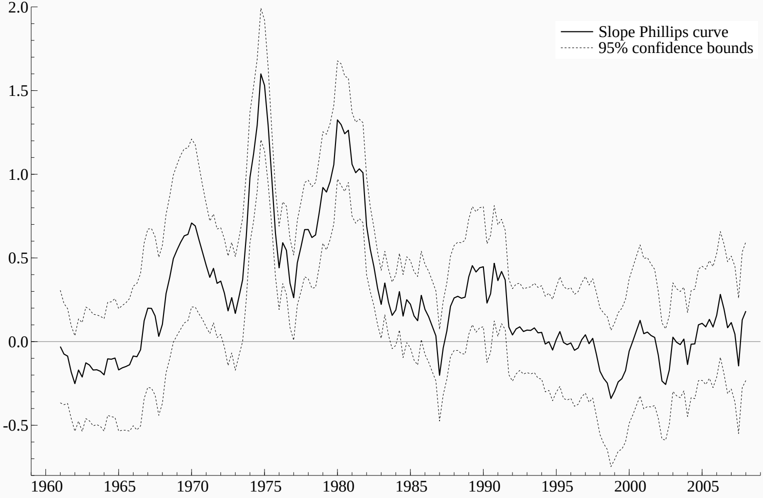
Initially, we consider the model of Ball & Mazumder (2012)

$$\pi_t = \beta_0 + \beta_1(\pi_{t-1} + \ldots + \pi_{t-4}) + \beta_{3,t} un_t + \varepsilon_t$$

There is one time-varying parameter

$$\beta_{3,t+1} = \beta_{3,t} + \eta_{3,t}$$

Slope of Phillips curve



The decline is clearly visible

Slope of Phillips curve

- A potential concern is that the other parameters of the model might also vary over time

We modify the model

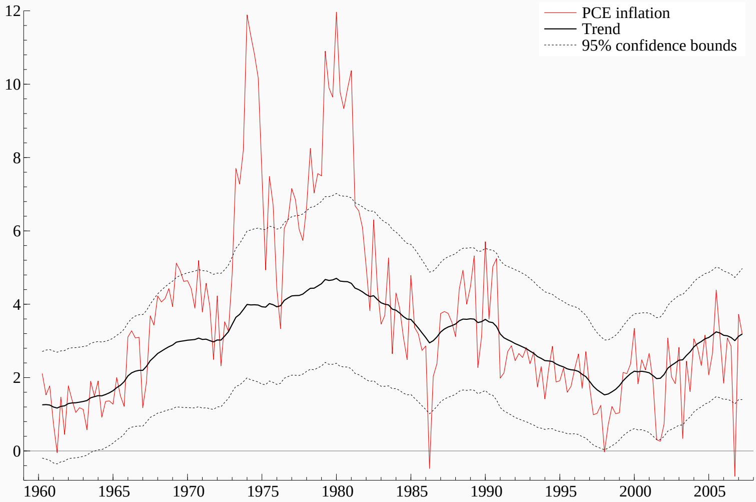
$$\pi_t = \beta_{0,t} + \beta_{1,t}(\pi_{t-1} + \ldots + \pi_{t-4}) + \beta_{3,t} un_t + \varepsilon_t$$

Now there are three time-varying parameters

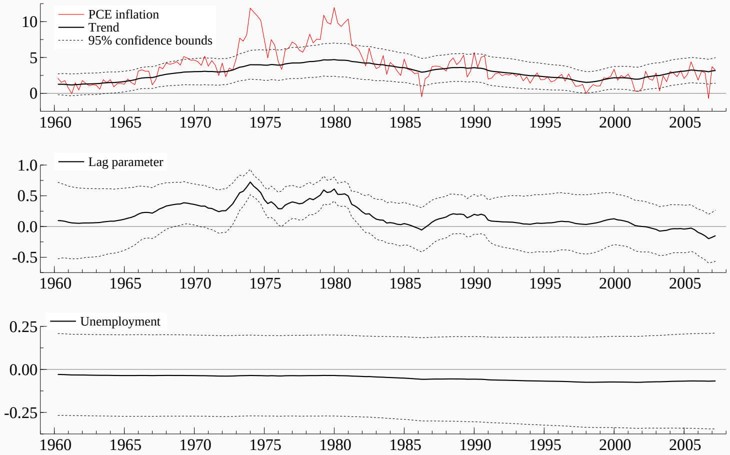
$$\boldsymbol{\beta}_t = \boldsymbol{\beta}_t + \boldsymbol{\eta}_t$$

with $\boldsymbol{\beta}_t = (\beta_{1,t}, \beta_{2,t}, \beta_{3,t})'$

Inflation trend



Slope of Phillips curve



Some comments

- The unemployment rate is endogenous, which implies that we only pick up correlations
- It would be better to combine this approach with an instrumental variable approach to project out confounding supply shocks
- This is challenging !!!

Final comments

State space models

- State space models form a very general model class which includes many models
- They are useful for measurement of latent components and forecasting

Reference & Material

- References:
  'Time Series Analysis by State Space Methods' by Jim Durbin and Siem Jan Koopman, Oxford University Press, 2012.